BER Analysis of M-QAM with Packet Combining over Space-Time Block Coded MIMO Fading Channels

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***Abstract*— We analyze the bit error rate (BER) performance of M-ary quadrature amplitude modulation (M-QAM) when using space-time block coding (STBC) along with packet combining triggered by automatic repeat request (ARQ) retransmission over multiple-input multiple-output (MIMO) fading channels. Specifically, adopting a log-likelihood ratio (LLR) based approach and considering the 16-QAM case of study, we provide an exact formulation for the aggregate LLR distribution in the case the STBC codeword can be transmitted twice, and derive the resulting BER. For higher number of retransmissions, an approximation of the error function is used to derive the LLR distributions and the system's ensuing BER. Considering different values of combined transmissions and M-QAM with possible constellation rearrangement (CoRe), validation of the proposed BER analytical model through simulations and assessment of the advantages of packet combining are provided for transmissions over additive white Gaussian noise (AWGN) channel and orthogonalized MIMO Rayleigh fading channels with different STBC mappings.**

***Index Terms*— Space-time block coding (STBC), ARQ, packet combining, BER, logarithmic likelihood ratio (LLR), multiple-input multiple-output (MIMO), Rayleigh fading.**

I. Introduction

DEVELOPMENT of transmission techniques aiming at improving the data rate and providing reliable communication over mobile networks has recently witnessed considerable interest especially with the advent of the multiple-input multiple-output (MIMO) technology. Indeed, since the work by Foschini [1] and Telatar [2], a lot of efforts were dedicated to the study of antenna array systems, both to evaluate the link-level performances and to present coding techniques suitable for the MIMO technology. One of the most simple and attractive techniques is space-time block coding (STBC), first proposed by Alamouti for two transmit antennas [3] and then extended to other MIMO configurations in [4]. In MIMO systems, the use of STBC in conjunction with link adaptation techniques can also allow coping with the varying nature of the propagation conditions to optimally exploit the available capacity of the channel in order to maximize throughput and satisfy delay and error rate requirements [5].

Link adaptation can be performed in various forms, for instance, in the form of automatic repeat request (ARQ) retransmission implemented at the link layer. In an ARQ protocol, link layer acknowledgements are used for retransmission decisions, whereby an erroneously received packet is eventually retransmitted until the packet is correctly received. In its simplest form, the ARQ protocol is of type I (ARQ-I) where each retransmitted packet carries the same information as in the original transmission [6]. ARQ-I is considered for future high-speed wireless networks, and is already adopted in systems such as High-Speed Downlink Packet Access (HSDPA) [7]. To reduce the delay effects of the protocol and avoid buffer overflow, the latter can be implemented in a truncated form, wherein the number of retransmissions is limited. Upon reception of a negative acknowledgement (NACK), sent by the receiver to the transmitter as a retransmission request over the reverse channel, retransmissions can continue until the packet is correctly received or a preset maximum number of retransmissions is attained. Mapping diversity, in the form of constellation rearrangement (CoRe) such as the one used in HSDPA [7], can also be used to improve the transmission reliability. The performance of ARQ-I protocol can further be enhanced through simple processing at the receiver. Indeed, instead of discarding packets that are received in error, packet combining can be implemented at the receiver [6], [8]. Using ARQ-I with packet combining, the number of retransmissions required for a correct packet reception can be reduced, thus increasing the system performance not only in terms of delay but also reliability and throughput efficiency.

Herein, we consider a MIMO STBC system employing M-ary quadrature amplitude modulation (M-QAM) and implementing ARQ-I with packet combining. Our study is intended at evaluating the bit error rate (BER) performance of the system when multiple received replicas of the STBC codeword are combined at the receiver. Accordingly, packet combining is also referred to as code combining and terms are used interchangeably throughout the paper.

The BER performance of MIMO STBC has previously been studied considering different modulation schemes and various channel models. Two main approaches can be used for this purpose. The first relies on converting the MIMO system model into an equivalent single-input single-output (SISO) model [9] [10], and the second is based on a log-likelihood ratio (LLR) approach [11]. Studies considering the combination of adaptive modulation and coding (AMC) with truncated ARQ for packet data transmission in MIMO-STBC systems have also been presented [5].

Compared to previous works, the main contribution of this paper is the performance analysis of MIMO-STBC when ARQ is used along with packet combining. More specifically, we combine both methods to present a general analysis for the BER performance of STBC used in conjunction with packet combining in MIMO Rayleigh fading channels. In the analysis, an LLR-based approach, which is transparent

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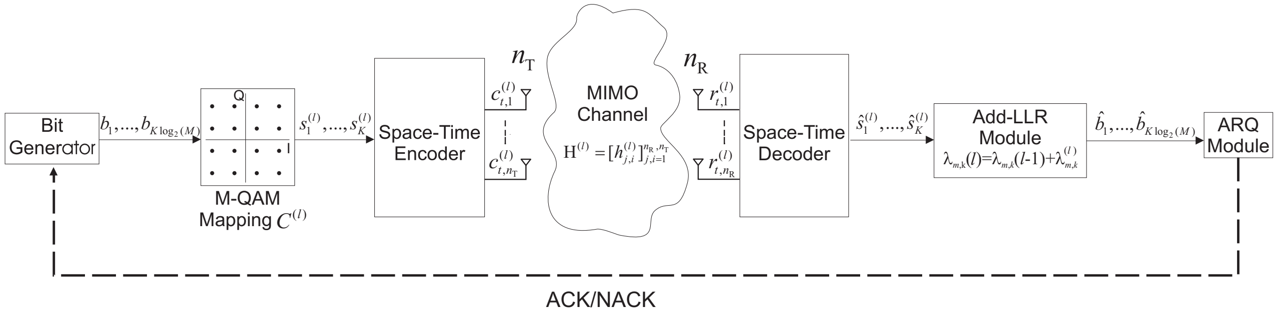

Fig. 1. System model.

to the modulation used in each transmission, is adopted. Herein, we consider M-QAM with fixed modulation levels among transmissions, and provide our analysis for $M = 16$ when using ARQ with packet combining. The latter is also considered in [12] where the focus is on the evaluation of the BER performance of 16-QAM in additive white Gaussian noise (AWGN) channel. Compared to the work therein, this letter considers a MIMO-STBC setting under Rayleigh fading and presents an exact analytical expression for the LLRs used in formulating the system's BER considering two combined M-QAM transmissions. Furthermore, by means of a simple approximation of the error function erf(.), we provide analytical formulation for the BER resulting from combining more than two transmissions.

The derived LLRs can be used for soft-input decoding in turbo-coded transmission as in HSDPA [7]. Moreover, even though the analysis is presented for 16-QAM, it can straightforwardly be extended to other M-QAM modulations, for coded transmission, and further elaborated for the analysis of the system under consideration when link adaptation, in the form of AMC and power adaptation, is used under partial channel information knowledge. The proposed BER analytical framework is corroborated through simulations for different STBC mappings, QAM with $M = 16$ or 32 and possible constellation rearrangement, and various values for the maximum number of allowed ARQ retransmissions.

The following content of the paper is structured into six sections. Section II presents the system and channel models. In Section III, we provide a general formulation for the bit LLRs. The probability density functions (PDF) of the aggregate bit LLRs are derived in Section IV, and then used in Section V to evaluate the system's BER. Comparisons of the analytical results with simulations are presented in Section VI, followed by conclusions drawn in Section VII.

II. System and Channel Models

We consider a wireless communication system where data transmission is organized in packets and ARQ is implemented to retransmit an erroneously received packet up to $L-1$ times. The communication system is equipped with n_{T} antenna elements at the transmitter and n_{R} antenna elements at the receiver as illustrated in Fig. 1, wherein the upper indexing $(.)^{(l)}$ is used to denote the parameters and signals pertaining to the l^{th} transmission with $l = 1, \cdots, L$. The MIMO channel is assumed to be a slowly-varying flat Rayleigh fading channel with the gain of the link between the i^{th} transmit and j^{th} receive antennas at transmission l, $h_{j,i}^{(l)}$, defined by a zero-mean circularly symmetric complex Gaussian random variable with unit power. In order to form a packet of K symbols, every $K \log_2 M$ information bits $\{b_m\}_{m=1}^{K \log_2 M}$ are mapped into symbols $\{s_k^{(l)}\}_{k=1}^{K}$ which are selected from a M-QAM constellation $\mathcal{C}^{(l)}$ with average energy E_0. Moreover, at each transmission l, a constellation rearrangement [7] can be utilized, hence the use of the parameter l in the constellation notation, i.e., $\mathcal{C}^{(l)}$. The resulting symbols $\{s_k^{(l)}\}_{k=1}^{K}$ are then encoded by a space-time block code defined by a $p \times n_{\mathrm{T}}$ columnwise orthogonal transmission matrix $\mathcal{G}_{n_{\mathrm{T}}}$. The entries of the resulting matrix $\mathcal{G}_{n_T}^{(l)} = [c_{t,i}^{(l)}]_{t,i=1}^{p,n_{\mathrm{T}}}$ are linear functions of $\{s_k^{(l)}\}_{k=1}^{K}$ and their conjugates. At each time slot t, $t = 1, \ldots, p$, the t^{th} row of $\mathcal{G}_{n_{\mathrm{T}}}^{(l)}$ is transmitted simultaneously through the n_{T} antennas. The symbol transmission rate R is defined as the ratio between the number of received symbols K and the STBC codeword duration p, i.e., $R = K/p$. For the l^{th} packet transmission, the signal received by the j^{th} antenna at time t, $t = 1, \ldots, p$, is given by

$$r_{t,j}^{(l)} = \sum_{i=1}^{n_{\mathrm{T}}} h_{j,i}^{(l)} c_{t,i}^{(l)} + n_{t,j}^{(l)}, \tag{1}$$

where $n_{t,j}^{(l)} \sim \mathcal{N}_c(0, N_0)$ is a zero-mean complex Gaussian noise with variance $N_0/2$ per dimension. The signal-to-noise ratio (SNR) per receive antenna is given by E_s/N_0, where E_s is the average power of the received signal at each receive antenna. Considering normalized average power over the transmit antennas, the average energy of the M-QAM constellation $\mathcal{C}^{(l)}$ is given by [9]

$$E_0 = \frac{(2M-2)}{3} d^2 = \frac{E_s}{n_{\mathrm{T}} a R}, \tag{2}$$

where $2d$ is the minimum distance between two symbols of $\mathcal{C}^{(l)}$, the same for every transmission l, and a is a constant which depends on the considered STBC matrix $\mathcal{G}_{n_{\mathrm{T}}}$ [9], [10]. Assuming that the channel coefficients are known at the receiver and using the SISO equivalency of the MIMO STBC model [9], [10], the received signal pertaining to the l^{th} packet transmission can be written as

$$\hat{r}_k^{(l)} = a \|\mathbf{H}^{(l)}\|_{\mathrm{F}}^2 s_k^{(l)} + \zeta_k^{(l)}, \qquad k = 1, \cdots, K, \tag{3}$$

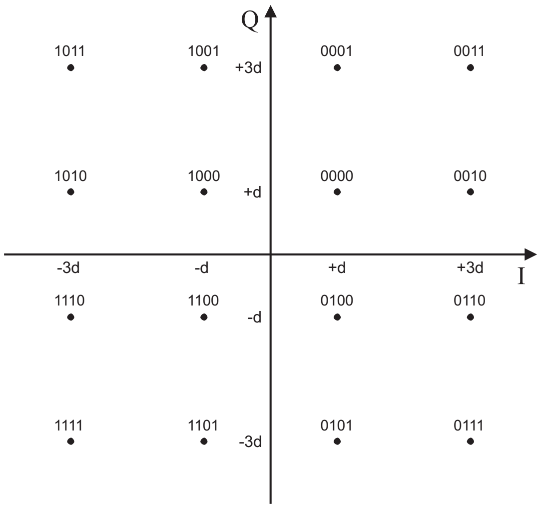

Fig. 2. 16-QAM constellation using Gray mapping.

where $\mathbf{H}^{(l)} = [h_{j,i}^{(l)}]_{j,i=1}^{n_\mathrm{R},n_\mathrm{T}}$, $\|\cdot\|_\mathrm{F}$ is the matrix Frobenius norm[1], and $\zeta_k^{(l)} \sim \mathcal{N}_c(0, a\|\mathbf{H}^{(l)}\|_\mathrm{F}^2 N_0)$. Equation (3) can be reformulated as

$$\hat{s}_k^{(l)} = s_k^{(l)} + \eta_k^{(l)}, \qquad k = 1, \cdots, K, \tag{4}$$

where $\hat{s}_k^{(l)} = \frac{\hat{r}_k^{(l)}}{a\|\mathbf{H}^{(l)}\|_\mathrm{F}^2}$ and $\eta_k^{(l)} \sim \mathcal{N}_c(0, \frac{N_0}{a\|\mathbf{H}^{(l)}\|_\mathrm{F}^2})$. As can be seen, the equivalent SISO model is a set of K parallel AWGN channels, each with noise variance $N_0^{(l)} = \frac{N_0}{a\|\mathbf{H}^{(l)}\|_\mathrm{F}^2}$. Finally, defining $\rho^{(l)} \triangleq \|\mathbf{H}^{(l)}\|_\mathrm{F}^2$ which follows a Gamma distribution, the instantaneous SNR per symbol at the l^th transmission is given by $\gamma^{(l)} = \rho^{(l)} E_s/(n_\mathrm{T} R N_0)$ and distributed according to the PDF

$$f_\gamma(\gamma^{(l)}) = \frac{(\gamma^{(l)})^{n_\mathrm{T} n_\mathrm{R}-1}}{\Gamma(n_\mathrm{T} n_\mathrm{R})\ \overline{\gamma}^{n_\mathrm{T} n_\mathrm{R}}} e^{-\gamma^{(l)}/\overline{\gamma}}, \tag{5}$$

where $\overline{\gamma} \triangleq E_s/(n_\mathrm{T} R N_0)$ and $\Gamma(\cdot)$ is the Gamma function.

In the following, even if the LLR model is presented considering the 16-QAM case of study and the mapping presented in Fig. 2, extension to other M-QAM modulations can easily be obtained following the procedure described hereafter.

III. Bit LLRs under Packet Combining

At transmission l, the LLR $\lambda_{m,k}^{(l)}$ corresponding to bit b_m, $m = 1, 2, 3, 4$, of symbol s_k, $k = 1, \ldots, K$, is defined as [11]

$$\begin{aligned}\lambda_{m,k}^{(l)} &= \log \frac{\Pr\{b_m = 1|\hat{s}_k, \mathbf{H}\}}{\Pr\{b_m = 0|\hat{s}_k, \mathbf{H}\}} \\ &= \log \frac{\sum_{c_1 \in S_m^{(1)}} \exp(-\|\hat{s}_k - c_1\|^2/\sigma_{k,l}^2)}{\sum_{c_0 \in S_m^{(0)}} \exp(-\|\hat{s}_k - c_0\|^2/\sigma_{k,l}^2)},\end{aligned} \tag{6}$$

where $S_m^{(0)}$ ($S_m^{(1)}$) is the group of symbols c_0 (c_1) with the m^th bit index equal to zero (one), $\sigma_{k,l}^2 = N_0$ in the case of AWGN channel, $\sigma_{k,l}^2 = N_0^{(l)}$ when considering MIMO Rayleigh fading channels, and for convenience we dropped the indexing $(.)^{(l)}$ from the data and channel matrix notations. Using the standard Max-Log approximation, $\log(\sum_i \exp(-X_i)) \approx -\min_i(X_i)$, the bit LLR (6) can be expressed as [11]

$$\lambda_{m,k}^{(l)} \approx \frac{1}{\sigma_{k,l}^2} \left(\min_{c_0 \in S_m^{(0)}} \|\hat{s}_k - c_0\|^2 - \min_{c_1 \in S_m^{(1)}} \|\hat{s}_k - c_1\|^2 \right). \tag{7}$$

Assuming Gray mapping, the signals in the phase-quadrature components can be perfectly separated at the demodulator. Moreover, results over the real (Inphase) and imaginary (Quadrature) dimensions being equivalent, we restrict our analysis to the bits corresponding to the real part of the transmitted symbols (bits at even positions). In this vein, the above bit LLR at positions $m = 2, 4$ can be reformulated as [11]

$$\lambda_{4,k}^{(l)} = \begin{cases} -4\psi_k^{(l)} \hat{s}_{kI} & \text{, if } |\hat{s}_{kI}| \le 2d \\ 8\psi_k^{(l)}(d - \hat{s}_{kI}) & \text{, if } \hat{s}_{kI} > 2d \\ -8\psi_k^{(l)}(d + \hat{s}_{kI}) & \text{, if } \hat{s}_{kI} < -2d \end{cases} \tag{8}$$

and,

$$\lambda_{2,k}^{(l)} = 4\psi_k^{(l)}(|\hat{s}_{kI}| - 2d), \tag{9}$$

where $\hat{s}_{kI} = \Re(\hat{s}_k)$ and $\psi_k^{(l)} = d/\sigma_{k,l}^2$.

Among various soft-output packet combining schemes, the add-LLR is known to be a simple method that does not increase the decoder complexity while being throughput-effective [8]. Assuming add-LLR combining, the aggregate LLR resulting from combining L transmissions of the same symbol is given by

$$\lambda_{m,k}(L) = \sum_{l=1}^{L} \lambda_{m,k}^{(l)}, \tag{10}$$

while when considering constellation rearrangement, it is expressed as

$$\lambda_{m,k}(L) = \sum_{l=1}^{L} \pm\lambda_{m^{(l)},k}^{(l)}, \tag{11}$$

where $m^{(l)}$ denotes the bit's position at transmission l of bit b_m initially transmitted at position m, and bit LLRs are either subtracted or added to the aggregate LLR depending on whether their signs were changed or not compared to the first transmission.

IV. Aggregate LLR Distributions

Bit LLRs for the single transmission case follow piecewise Gaussian distributions [12]. Indeed, the bit LLR PDF corresponding to index $m = 4$ of the transmitted symbols s_k, $k = 1, \ldots, K$, is given by

$$p_4(\lambda_{4,k}^{(l)}|s_{kI}, \psi_k^{(l)}) = \begin{cases} \frac{1}{\sqrt{\pi 16 d\psi_k^{(l)}}} \exp\left(-\frac{(\lambda_{4,k}^{(l)} + 4d\psi_k^{(l)} s_{kI})^2}{16 d\psi_k^{(l)}}\right) & \text{, if } |\lambda_{4,k}^{(l)}| \le 8d\psi_k^{(l)} \\ \frac{1}{\sqrt{\pi 64 d\psi_k^{(l)}}} \exp\left(-\frac{(\lambda_{4,k}^{(l)} + 8\psi_k^{(l)}(s_{kI} - d))^2}{64 d\psi_k^{(l)}}\right) & \text{, if } \lambda_{4,k}^{(l)} < -8d\psi_k^{(l)} \\ \frac{1}{\sqrt{\pi 64 d\psi_k^{(l)}}} \exp\left(-\frac{(\lambda_{4,k}^{(l)} + 8\psi_k^{(l)}(s_{kI} + d))^2}{64 d\psi_k^{(l)}}\right) & \text{, if } \lambda_{4,k}^{(l)} > 8d\psi_k^{(l)} \end{cases} \tag{12}$$

[1]The matrix Frobenius norm of an $r \times t$ matrix $\mathbf{M} = [m_{j,k}]_{j,k=1}^{r,t}$ is defined as $\|\mathbf{M}\|_F^2 = \sum_{j=1}^{r}\sum_{k=1}^{t} |m_{j,k}|^2 = \mathrm{trace}\left(\mathbf{M}\mathbf{M}^H\right)$, where $(.)^H$ stands for the transpose conjugate operator.

and the one corresponding to the bit with index $m = 2$, is given by

$$p_2(\lambda_{2,k}^{(l)}|s_{kI},\psi_k^{(l)}) = \begin{cases} \frac{1}{\sqrt{\pi 16 d\psi_k^{(l)}}}\left[\exp\left(\frac{(\lambda_{4,k}^{(l)}-4\psi_k^{(l)}(s_{kI}-2d))^2}{16d\psi_k^{(l)}}\right)+ \right. \\ \qquad \left. \exp\left(\frac{(\lambda_{4,k}^{(l)}+4\psi_k^{(l)}(s_{kI}+2d))^2}{16d\psi_k^{(l)}}\right)\right], & \text{if } \lambda_{2,k}^{(l)} \geq -8d\psi_k^{(l)} \\ 0 & \text{, if } \lambda_{2,k}^{(l)} < -8d\psi_k^{(l)}. \end{cases} \tag{13}$$

Considering independent channels from one transmission to another, the aggregate LLR distribution is equivalent to the convolution of individual LLR distributions defined for each transmission of the same symbol s_k. Moreover, since the effect of CoRe is a simple swapping of bits and/or modification of their signs [7], the LLR distributions remain Gaussian piecewise functions. Hence, analysis of the LLR distribution is quite the same with or without CoRe. Thus, to alleviate notations, we hereafter consider no CoRe. The aggregate bit LLR distribution after L transmissions is then given by

$$p_m(\lambda_{m,k}(L)|s_{kI}) = p_m(\lambda_{m,k}^{(1)}|s_{kI}) \ * \ldots * \ p_m(\lambda_{m,k}^{(L)}|s_{kI}), \tag{14}$$

where $*$ denotes the convolution operator.

To evaluate (14) and provide analytical expressions for the aggregate LLR PDFs, we introduce the function $f(z,\mu_1,\mu_2,\vartheta_1,\vartheta_2,r_2,r_1)$ defined in (15), where $\mathrm{erf}(\cdot)$ is the error function defined by $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2}\mathrm{d}x$. The function $f(z,\mu_1,\mu_2,\vartheta_1,\vartheta_2,r_2,r_1)$ can be seen as a *generalized convolution function*. Indeed, $f(z,\mu_1,\mu_2,\vartheta_1,\vartheta_2,+\infty,-\infty)$ is equivalent to the standard convolution of two Gaussian functions.

Considering two transmissions ($L = 2$) and using the definition of f, the PDFs of the aggregate LLRs $\lambda_{m,k}(2)$, $m = 2,4$, are shown in (16) and (17), where for convenience we used the following notations: $\mathbf{U} = \{u_{ij}\}_{i,j=1}^{4,3}$, $\mathbf{V} = \{v_{ij}\}_{i,j=1}^{3,2}$ and $\mathbf{W} = \{w_i\}_{i=1}^{2}$ with

$$\mathbf{U} = \begin{bmatrix} -8\psi_k^{(1)}(s_{kI}+d) & -4\psi_k^{(1)}s_{kI} & -8\psi_k^{(1)}(s_{kI}-d) \\ 64d\psi_k^{(1)} & 16d\psi_k^{(1)} & 64d\psi_k^{(1)} \\ -8\psi_k^{(2)}(s_{kI}+d) & -4\psi_k^{(2)}s_{kI} & -8\psi_k^{(2)}(s_{kI}-d) \\ 64d\psi_k^{(2)} & 16d\psi_k^{(2)} & 64d\psi_k^{(2)} \end{bmatrix},$$

$$\mathbf{V} = \begin{bmatrix} 4\psi_k^{(1)}(s_{kI}-2d) & -4\psi_k^{(1)}(s_{kI}+2d) \\ 4\psi_k^{(2)}(s_{kI}-2d) & -4\psi_k^{(2)}(s_{kI}+2d) \\ 16d\psi_k^{(1)} & 16d\psi_k^{(2)} \end{bmatrix},$$

$$\mathbf{W} = \begin{bmatrix} -8d\psi_k^{(1)} & -8d\psi_k^{(2)} \end{bmatrix},$$

and we assumed $w_1 > w_2$.

These PDFs are represented by piecewise functions formed by the sum of $f(z,\mu_1,\mu_2,\vartheta_1,\vartheta_2,r_2,r_1)$ terms. As it can be seen, the PDF formulae for $L = 2$ are more complicated compared to the Gaussian piecewise distributions, (12), (13), resulting when a single packet transmission is performed ($L = 1$). This complexity increases as the number of combined transmissions increases, which makes formulation of the LLR distributions in closed-form untractable. Indeed, due to the error function terms, erf($\cdot$), involved in the function $f(z,\mu_1,\mu_2,\vartheta_1,\vartheta_2,r_2,r_1)$, we need to compute untractable integrals of the form

$$\int e^{-x^2}\mathrm{erf}(ax+b)\mathrm{d}x. \tag{18}$$

$$\begin{aligned} f(z,\mu_1,\mu_2,\vartheta_1,\vartheta_2,r_2,r_1) &= \int_{r_1}^{r_2} \frac{1}{\pi\sqrt{\vartheta_1\vartheta_2}} e^{-\frac{(x-\mu_1)^2}{\vartheta_1}} e^{-\frac{(z-x-\mu_2)^2}{\vartheta_2}}\mathrm{d}x \\ &= \frac{1}{2\sqrt{\pi(\vartheta_1+\vartheta_2)}} e^{-\frac{(z-(\mu_1+\mu_2))^2}{\vartheta_1+\vartheta_2}} \left[\mathrm{erf}\left(\frac{\vartheta_1(r_2-z+\mu_2)+\vartheta_2(r_2-\mu_1)}{\sqrt{\vartheta_1\vartheta_2(\vartheta_1+\vartheta_2)}}\right) - \mathrm{erf}\left(\frac{\vartheta_1(r_1-z+\mu_2)+\vartheta_2(r_1-\mu_1)}{\sqrt{\vartheta_1\vartheta_2(\vartheta_1+\vartheta_2)}}\right)\right]. \end{aligned} \tag{15}$$

$$p_4(\lambda_{4,k}(2)|s_{kI},\psi_k^{(1)},\psi_k^{(2)}) =$$
$$\begin{cases} f(\lambda_{4,k}(2),u_{13},u_{31},u_{23},u_{41},\lambda_{4k}(2)-w_2,-\infty) + f(\lambda_{4,k}(2),u_{13},u_{32},u_{23},u_{42},\lambda_{4,k}(2)+w_2,\lambda_{4,k}(2)-w_2) \\ +f(\lambda_{4,k}(2),u_{13},u_{33},u_{23},u_{43},-w_1,\lambda_{4,k}(2)+w_2) + f(\lambda_{4,k}(2),u_{12},u_{33},u_{22},u_{43},w_1,-w_1) \\ +f(\lambda_{4,k}(2),u_{11},u_{33},u_{21},u_{43},+\infty,w_1) & \text{, if } \lambda_{4,k}(2) \leq -(w_1+w_2) \\ f(\lambda_{4,k}(2),u_{13},u_{31},u_{23},u_{41},\lambda_{4,k}(2)-w_2,-\infty) + f(\lambda_{4,k}(2),u_{13},u_{32},u_{23},u_{42},-w_1,\lambda_{4,k}(2)-w_2) \\ +f(\lambda_{4,k}(2),u_{12},u_{32},u_{22},u_{42},\lambda_{4,k}(2)+w_2,-w_1) + f(\lambda_{4,k}(2),u_{12},u_{33},u_{22},u_{43},w_1,\lambda_{4,k}(2)+w_2) \\ +\ f(\lambda_{4,k}(2),u_{11},u_{33},u_{21},u_{4,3},+\infty,w_1) & \text{, if } -(w_1+w_2) < \lambda_{4,k}(2) \leq -(w_1-w_2) \\ f(\lambda_{4,k}(2),u_{1,3},u_{3,1},u_{2,3},u_{4,1},-w_1,-\infty) + f(\lambda_{4,k}(2),u_{1,2},u_{3,1},u_{2,2},u_{4,1},\lambda_{4,k}(2)-w_2,-w_1) \\ +f(\lambda_{4,k}(2),u_{12},u_{32},u_{22},u_{42},\lambda_{4,k}(2)+w_2,\lambda_{4,k}(2)-w_2) + f(\lambda_{4,k}(2),u_{12},u_{33},u_{22},u_{43},w_1,\lambda_{4,k}(2)+w_2) \\ +f(\lambda_{4,k}(2),u_{11},u_{33},u_{21},u_{43},+\infty,w_1) & \text{, if } -(w_1-w_2) < \lambda_{4,k}(2) \leq (w_1-w_2) \\ f(\lambda_{4,k}(2),u_{13},u_{31},u_{23},u_{41},-w_1,-\infty) + f(\lambda_{4,k}(2),u_{12},u_{31},u_{22},u_{41},\lambda_{4,k}(2)-w_2,-w_1) \\ +f(\lambda_{4,k}(2),u_{12},u_{32},u_{22},u_{42},w_1,\lambda_{4,k}(2)-w_2) + f(\lambda_{4,k}(2),u_{11},u_{32},u_{21},u_{42},\lambda_{4,k}(2)+w_2,w_1) \\ +f(\lambda_{4,k}(2),u_{11},u_{33},u_{21},u_{43},+\infty,\lambda_{4,k}(2)+w_2) & \text{, if } (w_1-w_2) < \lambda_{4,k}(2) \leq (w_1+w_2) \\ f(\lambda_{4,k}(2),u_{13},u_{31},u_{23},u_{41},-w_1,-\infty) + f(\lambda_{4,k}(2),u_{12},u_{31},u_{22},u_{41},w_1,-w_1) \\ +f(\lambda_{4,k}(2),u_{11},u_{31},u_{21},u_{41},\lambda_{4,k}(2)-w_2,w_1) + f(\lambda_{4,k}(2),u_{12},u_{31},u_{22},u_{41},\lambda_{4,k}(2)+w_2,\lambda_{4,k}(2)-w_2) \\ +f(\lambda_{4,k}(2),u_{11},u_{33},u_{21},u_{43},+\infty,\lambda_{4,k}(2)+w_2) & \text{, if } (w_1+w_2) < \lambda_{4,k}(2). \end{cases} \tag{16}$$

$$p_2(\lambda_{2,k}(2)|s_{kI},\psi_k^{(1)},\psi_k^{(2)}) =$$
$$\begin{cases} f(\lambda_{2,k}(2),v_{11},v_{21},v_{31},v_{32},\lambda_{2,k}(2)+w_2,-w_1) + f(\lambda_{2,k}(2),v_{11},v_{22},v_{31},v_{32},\lambda_{2,k}(2)+w_2,-w_1) \\ +f(\lambda_{2,k}(2),v_{12},v_{21},v_{31},v_{32},\lambda_{2,k}(2)+w_2,-w_1) + f(\lambda_{2,k}(2),v_{12},v_{22},v_{31},v_{32},\lambda_{2,k}(2)+w_2,-w_1) \\ & \text{, if } -(w_1+w_2) \leq \lambda_{2,k}(2) \\ 0 & \text{, if } \lambda_{2,k}(2) < -(w_1+w_2). \end{cases} \tag{17}$$

To tackle this problem, we use a simple approximation for the Gaussian distribution function [13] to express the error function erf(·) as piecewise polynomial function according to:

$$\mathrm{erf}(x) \approx \begin{cases} -(0.4x^2 - 1.24|x|)\ \mathrm{sign}(x) & \text{if } |x| \le 1.556, \\ 0.98\ \mathrm{sign}(x) & \text{if } 1.556 < |x| < 1.838, \\ \mathrm{sign}(x) & \text{if } 1.838 \le |x|, \end{cases} \tag{19}$$

where $|\cdot|$ denotes the absolute value operator.

By means of this approximation, the aggregate LLR distributions considering two transmissions ($L = 2$) can be expressed in closed-form as the sum of piecewise functions of the form $q_j(x) = x^j e^{-(\alpha x+\beta)^2}$, $j \in \mathbb{N}$, where α and β are functions of s_{kI}, d and $\left\{\psi_k^{(1)}, \ldots, \psi_k^{(L)}\right\}$. Now, in order to compute the aggregate LLR distributions (14) when three transmissions ($L = 3$) are used, namely, $p_m(\lambda_{m,k}(3)|s_{kI}, \psi_k^{(1)}, \psi_k^{(2)}, \psi_k^{(3)})$, $m = 2, 4$, the primitive of $q_j(x)$ is needed. The latter is given by

$$Q_j(x) = \int x^j e^{-(\alpha x+\beta)^2} \mathrm{d}x, \tag{20}$$

where α and β are functions of s_{kI}, d and $\left\{\psi_k^{(1)}, \ldots, \psi_k^{(L)}\right\}$. Using integration by parts, $Q_j(x)$ can be expressed as

$$Q_j(x) = P_j(x)\ e^{-(\alpha x+\beta)^2} + C_j\ \mathrm{erf}(\alpha x + \beta), \tag{21}$$

where $P_j(x)$ is a polynomial function of degree $(j-1)$, except for $j = 0$ in which case it is equal to zero, and C_j is a constant independent of x. Indeed, using the following expressions:

$$Q_0(x) = \frac{\sqrt{\pi}\mathrm{erf}(\alpha x+\beta)}{2\alpha},$$
$$Q_1(x) = -\frac{1}{2\alpha^2} e^{-(\alpha x+\beta)^2} - \frac{\beta\sqrt{\pi}}{2\alpha^2}\mathrm{erf}(\alpha x+\beta),$$
$$2\alpha^2 Q_{j+2}(x) = (j+1)Q_j(x) - 2\alpha\beta Q_{j+1}(x) - x^{j+1}e^{-(\alpha x+\beta)^2},$$

we can easily show that

$$2\alpha^2 P_{j+2}(x) = (j+1)P_j(x) - 2\alpha\beta P_{j+1}(x) - x^{j+1}, \tag{22}$$
$$2\alpha^2 C_{j+2} = (j+1)C_j - 2\alpha\beta C_{j+1}, \tag{23}$$

with,

$$P_0 = 0 \quad , \quad P_1 = -1/2\alpha^2, \qquad C_0 = \sqrt{\pi}/2\alpha \quad , \quad C_1 = -\beta\sqrt{\pi}/2\alpha^2. \tag{24}$$

The aggregate LLR distributions for $L = 3$ are piecewise functions composed by the sum of terms of the form $Q_j(x)$. Thus, using (21) jointly with the definitions of $P_j(x)$ and $C_j(x)$, an analytical formulation of the PDFs $p_m(\lambda_{m,k}(3)|s_{kI}, \psi_k^{(1)}, \psi_k^{(2)}, \psi_k^{(3)})$, $m = 2, 4$, can easily be obtained. The LLR distributions assuming more than three transmissions can subsequently be deduced. In fact, using (19), the expression in (21) becomes equivalent to a sum of terms in the form of $q_j(x)$. Therefore, the same analysis used for $L = 3$ can be applied for the general case in order to provide analytical formulation of the aggregate LLR PDFs $p_m(\lambda_{m,k}(L)|s_{kI}, \psi_k^{(1)}, \ldots, \psi_k^{(L)})$, $m = 2, 4$, for L combined transmissions.

V. BER Derivation

In the case of AWGN channel, we have $\psi_k^{(l)} = d/N_0$ for $l = 1, \cdots, L$. Hence, the aggregate LLR distributions $p_m(\lambda_{m,k}(L)|s_k, \psi_k^{(1)}, \ldots, \psi_k^{(L)})$, $m = 1, \cdots, \log_2 M$, can be denoted by $p_m(\lambda_{m,k}(L)|s_k)$. The BER considering M-QAM and using L transmissions can then be expressed as

$$P_{\mathrm{e}}^{\mathrm{AWGN},L} = \frac{1}{\log_2 M} \sum_{m=1}^{\log_2 M} P_{\mathrm{e}}^{\mathrm{AWGN},L}(m), \tag{25}$$

where

$$P_{\mathrm{e}}^{\mathrm{AWGN},L}(m) = \frac{1}{M}\left[\sum_{s_k \in S_m^{(1)}} \int_{-\infty}^{0} p_m(\lambda_{m,k}(L)|s_k)\mathrm{d}\lambda_{m,k}(L) + \sum_{s_k \in S_m^{(0)}} \int_{0}^{\infty} p_m(\lambda_{m,k}(L)|s_k)\mathrm{d}\lambda_{m,k}(L)\right]. \tag{26}$$

Taking into consideration the fact that the LLR PDFs for L combined transmissions are piecewise functions defined by the sum of terms of the form $q_j(x)$, the analytical expression for the BER can then be formulated using (19)-(24) and following the procedure described in Section IV. On the other hand, when using STBC in MIMO configurations over Rayleigh fading channels, the parameter $d\,\psi_k^{(l)}$ is given by

$$d\,\psi_k^{(l)} = \frac{d^2}{N_0^{(l)}} = \frac{E_s\rho^{(l)}}{\phi\, n_{\mathrm{T}}\, R\, N_0} = \frac{\gamma^{(l)}}{\phi}, \tag{27}$$

where ϕ is a modulation-dependent constant. For instance, $\phi = 10$ in the 16-QAM case (2). Accordingly, the LLR distributions $p_m(\lambda_{m,k}(L)|s_k, \psi_k^{(1)}, \ldots, \psi_k^{(L)})$ can be denoted as $p_m(\lambda_{m,k}(L)|s_k, \gamma^{(1)}, \ldots, \gamma^{(L)})$. Using the derived PDFs, the general formulation of the BER for the STBC system with packet combining in MIMO Rayleigh fading channels, $P_{\mathrm{e}}^{\mathrm{STBC},L}$, can then be deduced using (5) and (25) in the following expression

$$P_{\mathrm{e}}^{\mathrm{STBC},L} = \int_0^{\infty} \cdots \int_0^{\infty} P_{\mathrm{e}}^{\mathrm{AWGN},L} f_\gamma(\gamma^{(1)}) \ldots f_\gamma(\gamma^{(L)}) \mathrm{d}\gamma^{(1)} \ldots \mathrm{d}\gamma^{(L)}. \tag{28}$$

VI. Numerical Results

Sample numerical results are now presented to illustrate the performance of the system under study. BER results are plotted as a function of the SNR per receive antenna, E_s/N_0, and comparisons between analytical and simulation results are carried out. For the AWGN channel case, analytical results are computed through direct use of (25) and (26). As for the STBC case, in addition to the latter expressions, we use the Gauss-Legendre Quadrature rule [14] for the evaluation of the integral involved in the BER computation (28). For comparison purposes, results pertaining to the reference cases, with no code combining ($L = 1$), are also provided.

Fig. 3 illustrates the BER performance when packet combining is used for up to $L = 4$ transmissions in an AWGN channel with and without CoRe. The rearrangement used prior

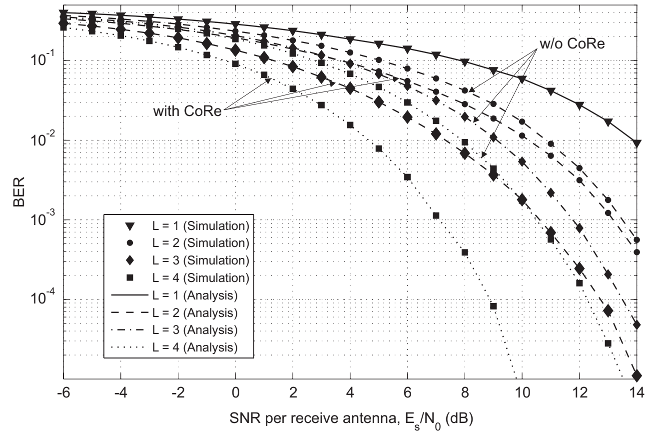

Fig. 3. Comparison between analytical and simulation results: 16-QAM used with or without CoRe and up to $L = 4$ transmissions over AWGN channel.

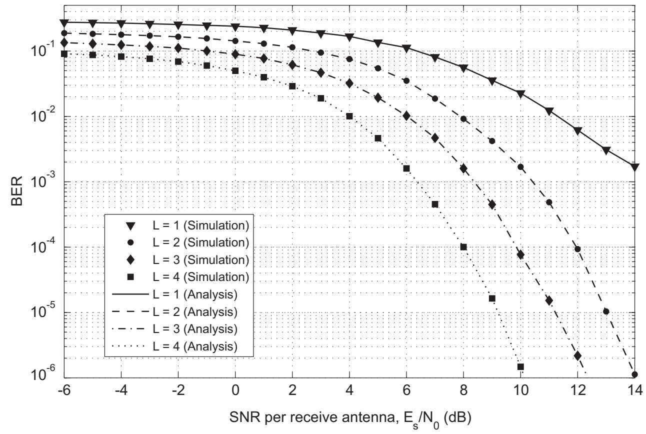

Fig. 5. BER performance when using STBC in 2×2 Rayleigh fading channels with 16-QAM, CoRe, and up to $L = 4$ combined transmissions.

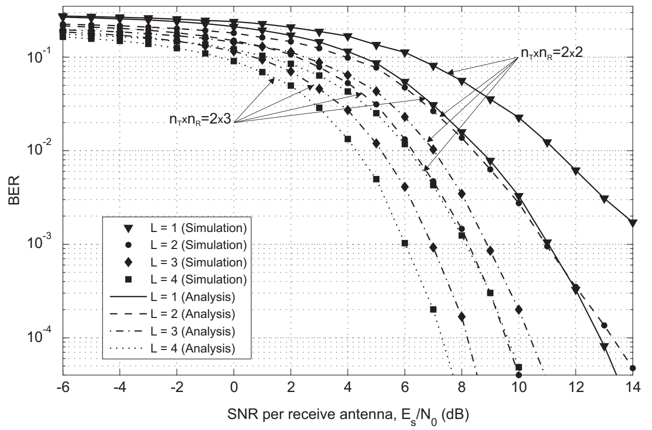

Fig. 4. BER performance when using STBC with 16-QAM and up to $L = 4$ combined transmissions in MIMO Rayleigh fading channels, code $\mathcal{G}_2$ and n_{R}=2 or 3.

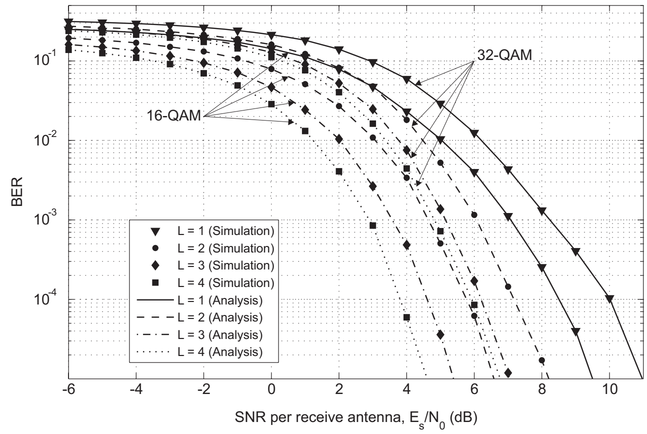

Fig. 6. BER performance when using STBC in 3×3 Rayleigh fading channels, code $\mathcal{G}_3$, QAM with $M = 16$ or 32, and up to $L = 4$ combined transmissions.

to symbol modulation with 16-QAM (Fig. 2) corresponds to the one provided in [7]. As observed in both cases, analytical and simulation results are in good agreement, which illustrates the validity of the proposed analytical modeling for the LLR PDFs and the accuracy of the approximation used for the error function. The plots also illustrate the gain obtained through CoRe.

We now consider the MIMO-STBC case and present the BER performance for different STBC mappings and system parameters for the purpose of illustrating the validity of our approach and the advantage of packet combining. Consider first the MIMO-STBC setting using the Alamouti code [3], namely, $\mathcal{G}_2$ with $a = 1$ and $R = 1$, and 16-QAM modulation with no CoRe. The BER performance is shown in Fig. 4 where MIMO configurations with $n_{\mathrm{T}} = 2$ and the value of n_{R} set to 2 or 3 are considered. Similar to the AWGN case, we observe that for different numbers of combined transmissions, the analytical results are almost in perfect match with the simulations, indicating the validity of the analytical model and the accuracy of the approximation used for the erf(.) function when $L > 2$. A closer look to Fig. 4 also reveals the advantage of packet combining in reducing the number of receive antennas required to achieve a target BER. For instance, at $\gamma = 10$dB, $L = 4$ transmissions are needed to yield a BER of 4×10^{-5} when $n_{\mathrm{T}} \times n_{\mathrm{R}} = 2 \times 2$, whereas the same BER target can be obtained when using the 2×3 MIMO configuration with $L = 2$. This illustrates that implementing packet combining can allow one to achieve a target BER performance while reducing the complexity in terms of the number of receive antennas required. Here it is worthwhile mentioning that the focus of the paper being not on providing design guidelines for the system parameters required to achieve target BERs for specific applications, delay/throughput/complexity tradeoffs will not be discussed herein. Remaining along the objective of this work, we consider the 2×2 MIMO configuration and illustrate in Fig. 5 the gain achieved when implementing CoRe. Examining the plots of the latter figure with those shown in Fig. 4 clearly illustrates how bit arrangement prior to signal modulation allows reducing the number of retransmissions required to achieve a target BER. For instance, at $\gamma = 10$dB, using $L = 3$ packet transmissions yield a BER of 8×10^{-5} whereas without CoRe one additional transmission would be needed to obtain a BER of 4×10^{-5}.

Finally, to illustrate the validity of our analytical model for other M-QAM modulations, we make use of the bit LLR expressions provided in [11] and present the BER performance when considering 32-QAM in a MIMO-STBC setting implementing the mapping provided in [4], namely, the half-rate code $\mathcal{G}_3$ with $a = 2$ and $R = 1/2$. This mapping is also used for the 16-QAM case and analytical results are corroborated through simulations as observed in Fig. 6, showing the BER performance for the 3×3 configuration with different values of combined transmissions.

VII. Conclusion

We considered a multi-element antenna system utilizing space-time block coding, and adopted a log-likelihood ratio (LLR) approach to present a BER analysis when M-QAM modulation is used along with code combining. Considering that the LLRs follow Gaussian piecewise distributions in AWGN channel and using a simple approximation for the error function $\text{erf}(\cdot)$, we presented an analytical formulation for the LLR distributions when packet retransmission in the form of ARQ protocol is used along with packet combining. The analysis considers 16-QAM used with or without constellation rearrangement, is performed for AWGN and MIMO Rayleigh fading channels, and can straightforwardly be extended to other M-QAM modulations. BER numerical results are provided for different system settings and parameters, and show the accuracy of the proposed analytical model and the advantages of packet combining.

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